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A Theoretical Study of the Generation of Atmospheric-Clear Air Turbulence

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This paper concerns a numerical study of the time history of the growth and decay of turbulence in the atmosphere. The study was motivated by a desire to better understand the behaviour of shear-generated turbulence in the Earth's atmosphere—the problem of clear air turbulence (CAT). The variant modeling technique is used to derive a closed set of equations that govern this phenomenon. The resulting set of equations allows, for the first time, a step-by-step numerical calculation of the growth of turbulent disturbances in atmospheric shear flows and the dependence of this growth on atmospheric instabilities.

Nomenclature§

A, B, C	= const
a_2, a_3	= const
c, c_1, c_2, c_3, c_4	= const
c_p	= specific heat at constant pressure
f	= unknown function
g	= acceleration due to gravity
g_{ij}	= the metric tensor
K	= $\langle u^{m\prime} u_m{}' \rangle$
k	= conductivity
k_T	= eddy conductivity
l	= Prandtl's mixing length
p	= pressure
q	= $K^{1/2} = \langle u^{m\prime} u_m{}' \rangle^{1/2}$
q_i	= heat-flow vector
Re_Λ	= $\rho \Lambda q / \mu$
T	= temperature
t	= time
u, v, w	= x, y, and z components of velocity
u_i	= velocity vector
X	= body force
x, y, z	= Cartesian coordinates
x_i	= general coordinate system
δ	= boundary-layer thickness or scale of mean motion
$\delta_i{}^j$	= Kronecker delta
ε_{ij}	= $\bar{u}_{i,j} + \bar{u}_{j,i}$
η	= function defined in Eq. (66)
λ, Λ	= scalar measures of length
μ, μ^*	= first and second coefficients of viscosity
μ_T	= eddy viscosity
ξ	= function defined in Eq. (64)
ρ	= density
σ	= $(1/\rho_0)\, \partial \rho_0 / \partial z$
τ_{ij}	= stress tensor
ϕ	= gravitational potential
ω	= $2[(g/T_0)(\partial \bar{T}/\partial z)]^{1/2}$

Subscript

0	= equilibrium conditions of the atmosphere

Superscripts

$'$	= departure from equilibrium
$\bar{\ }$	= mean
$'$	= departure from the mean

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§ The notation of general tensor analysis is used. In particular, the summation convention is implied, and a comma preceding a subscript indicates covariant differentiation.

Introduction

FOR many years, those engaged in the study of turbulent motions have made use of the concepts of eddy viscosity and eddy conductivity to enable them to calculate the conduction of momentum and heat in these motions. In an incompressible medium to which these notions have been extensively applied, the eddy-conductivity concept relates the turbulent flux of momentum and heat to the gradient of the mean velocity and temperature fields according to

$$\tau_{ij} = \mu(\bar{u}_{i,j}+\bar{u}_{j,i})-\rho\langle u_i'u_j'\rangle = (\mu+\mu_T)(\bar{u}_{i,j}+\bar{u}_{j,i}) \tag{1}$$

and

$$-q_i = k\bar{T}_{,i}-\rho c_p\langle u_i'T'\rangle = (k+k_T)\bar{T}_{,i} \tag{2}$$

The eddy viscosity μ_T and the eddy diffusivity k_T in these expressions have, in the past, been determined either by resort to experiment or, more indirectly, from physical arguments which determined some basic and perhaps simpler parameters which could in turn be evaluated experimentally. The latter method has, in general, been preferred because of the hope that such parameters might have some simple general relationships by which they could be determined for a large class of flows. Of course, both these methods fail if τ_{ij} or q_i are not zero when $\bar{u}_{i,j}+\bar{u}_{j,i}$ and $\bar{T}_{,i}$ vanish.

In view of the fact that the tensor $\langle u_i'u_j'\rangle$ and the vector $\langle u_i'T'\rangle$ both have dynamics of their own quite apart from (although related to) the dynamics of the mean fields $\bar{u}_i$ and T, the concept of eddy transport has been remarkably successful over the years in dealing with so many engineering problems. The explanation for this success lies in the fact that the dynamics of the turbulence is fast enough relative to the dynamics of the mean flow in many cases so that the turbulent correlations $\langle u_i'u_j'\rangle$ and $\langle u_i'T'\rangle$ are able to "track" the development of the mean motion. This is certainly true of developed pipe flows where the derivatives of all mean quantities, except the static pressure, with respect to streamwise positon are zero.

On the other hand, a number of turbulent flows of interest exist in nature for which the turbulent correlations are not able to track the changes in the mean field $\bar{u}_i$ and $\bar{T}$. In such flows, the dynamics of the turbulence is important, and the concept of eddy transport in any simple form is not a sufficiently powerful tool with which to study the motion.

As will be seen in what follows, the motion of the atmosphere is an example of such a flow. For this reason, a more powerful approach must be used—one which actually keeps track in some way of the dynamics of the turbulent correlations such as $\langle u_i'u_j'\rangle$ and $\langle u_i'T'\rangle$.

Within the past few years, a method has been developed by the senior author of this paper and his collaborators[1,2] which will, when it has been fully developed, permit one to calculate, with some degree of precision, the motion of strongly sheared turbulent layers. The method, which is called the method of invariant modeling, has been applied with some success to the calculation of incompressible turbulent boundary layers and to the decay of an isolated vortex. In this paper, we present the results of an initial application of the method to the calculation of atmospheric motions.

Brief Description of the Method of Invariant Modeling

It is beyond the scope of this paper to present a complete description of the method of invariant modeling. However, we give, in this section, a brief outline of the method, as well as a short discussion of its relationship to the mixing length model of turbulent transport proposed by Prandtl.[3]

Consider the motion of an incompressible medium which is governed by the Navier-Stokes equations so that the momentum equation may be written

$$\partial(\rho u_i)/\partial t+(\rho u^j u_i)_{,j} = -p_{,i}+\tau^j_{i,j} \tag{3}$$

where

$$\tau_{ij} = \mu(u_{i,j}+u_{j,i}) \tag{4}$$

If we write the velocity vector u_i and the pressure as the sum of a mean part and a fluctuating part whose time average is zero, i.e., $u_i = \bar{u}_i+u_i'$ and $p = \bar{p}+p'$ with $\bar{u}_i' = \bar{p}' = 0$, Eq. (3) can be written, after time-averaging, as

$$\frac{\partial}{\partial t}(\rho\bar{u}_i)+(\rho\bar{u}^j\bar{u}_i)_{,j} = -\bar{p}_{,i}+(\tau_i^j-\rho\langle u^{j\prime}u_i'\rangle)_{,j} \tag{5}$$

This is the turbulent momentum equation derived by Reynolds.

To solve this equation we need to know the Reynolds stress tensor $-\rho\langle u_i'u_j'\rangle$. If the turbulent correlations can track the mean motion in the sense discussed in the previous section, we can assume that this tensor can be expressed in terms of the mean velocity field $\bar{u}_i$.

Prandtl accomplished this for the case of a simple boundary-layer flow by means of a physical argument[3] and obtained the well-known result

$$-\rho\langle u'v'\rangle = \rho l^2\,|\partial\bar{u}/\partial y|\,\partial\bar{u}/\partial y \tag{6}$$

where l (the "mixing length") is a scale parameter identified in the argument with a characteristic size of the turbulent motion.

We can obtain this result in another way. We ask whether one can write the tensor $\langle u_i'u_j'\rangle$ in terms of the mean motion $\bar{u}_i$. The expression chosen must exhibit the symmetric-tensor character of $\langle u_i'u_j'\rangle$, must be dimensionally correct, must be invariant under a general transformation of coordinates, as well as under a Galilean transformation, and must not be such as to upset any of the general conservation laws. Since

$$\varepsilon_{ij} = \bar{u}_{i,j}+\bar{u}_{j,i} \tag{7}$$

has the required symmetry and satisfies the invariance conditions, one might write

$$-\langle u_i'u_j'\rangle = f(\bar{u}_k)(\bar{u}_{i,j}+\bar{u}_{j,i}) \tag{8}$$

where $f(\bar{u}_k)$ must be a scalar and must be such that the dimensions of the resulting expression are correct. Two simple choices for $f(\bar{u}_k)$ have the required Galilean invariance. They are

$$f_1(\bar{u}_k) = \Lambda^2\bar{u}_{,m}{}^m \tag{9}$$

and

$$f_2(\bar{u}_k) = \Lambda^2(\varepsilon_{mn}\varepsilon^{mn})^{1/2} \tag{10}$$

In these expressions, Λ is a scalar measure of length. The first expression above is zero by virtue of continuity, so we choose the second expression [Eq. (10)] so that

$$-\langle u_i'u_j'\rangle = \Lambda^2(\varepsilon_{mn}\varepsilon^{mn})^{1/2}(\bar{u}_{i,j}+\bar{u}_{j,i}) \tag{11}$$

If the formula given above is reduced to the situation of a simple shear layer $\bar{u} = \bar{u}(y)$ with $\bar{v} = \bar{w} = 0$, one obtains

$$-\langle u'v'\rangle = (2)^{1/2}\Lambda^2\,|\partial\bar{u}/\partial y|\,\partial\bar{u}/\partial y \tag{12}$$

which is essentially Prandtl's result. We conclude, therefore, that Eq. (11) represents a general form of Prandtl's mixing length expression for the Reynolds stress. We might expect such a result to hold provided the mean flow was such that the dynamics of the turbulence could track the mean motion. We also expect the scale parameter Λ to be related to the scale of the mean motion.

Suppose now we wish to generalize this idea to the case where the dynamics of the turbulence must be considered. We can write the equation for the Reynolds stress, namely,

$$\rho\frac{\partial\langle u_i'u_j'\rangle}{\partial t}+\rho\bar{u}^m\langle u_i'u_j'\rangle_{,m} = -\rho\langle u^{m\prime}u_j'\rangle\bar{u}_{i,m}-\rho\langle u^{m\prime}u_i'\rangle\bar{u}_{j,m}-$$
$$\rho\langle u^{m\prime}u_i'u_j'\rangle_{,m}-\langle u_i'p'\rangle_{,j}-\langle u_j'p'\rangle_{,i}+\langle p'(u_{i,j}'+u_{j,i}')\rangle+$$
$$\mu g^{mn}\langle u_i'u_j'\rangle_{,mn}-2\mu g^{mn}\langle u_{i,m}'u_{j,n}'\rangle \tag{13}$$

and we note that correlations other than the second-order correlation $\langle u_i'u_k'\rangle$ itself appear in the equation.

If we wish to close the set of equations to be solved with no more equations than those describing $\bar{u}_i$ and $\langle u_i'u_j'\rangle$, we must express these extra terms through the use of quantities for which we already have equations. In the method of invariant modeling, we do this by choosing expressions for these terms based on the second-order correlations themselves which satisfy the general rules for modeling set forth above in the discussion of Prandtl's mixing length. When this procedure is followed, one can actually obtain only a few models which satisfy the rules set down and which are reasonably simple expressions representing straightforward physical principles. The following model has been investigated in some detail:

$$\langle u^{m\prime}u_i'u_j'\rangle = -\Lambda q[g^{mn}\langle u_i'u_j'\rangle_{,n}+\langle u^{m\prime}u_i'\rangle_{,j}+\langle u^{m\prime}u_j'\rangle_{,i}] \tag{14}$$

$$\langle p'(u'_{i,j}+u'_{j,i})\rangle = (\rho q/\Lambda)[g_{ij}(K/3)-\langle u_i'u_j'\rangle] \tag{15}$$

$$g^{mn}\langle u'_{i,m}u'_{j,n}\rangle = \langle u_i'u_j'\rangle/\lambda^2 \tag{16}$$

$$\langle u_i'p'\rangle = -\rho\Lambda q\langle u^{m\prime}u_i'\rangle_{,m} \tag{17}$$

In this model there are two scale parameters Λ and λ. They are related in concept to the integral and microscales of the turbulence. In our work, we have taken them to be related to each other by the expression

$$\lambda^2 = \Lambda^2/(c_1+c_2\,Re_\Lambda) \tag{18}$$

Equation (18) has been used by Glushkò[4] and by Beckwith and Bushnell[5] when modeling the turbulent kinetic energy equation obtained from Eq. (13) by contraction.

In applying the model to the incompressible boundary layer on a flat plate, we assume that in the outer regions of the boundary layer the scale Λ is a constant fraction of the boundary-layer thickness δ

$$\Lambda = c_3\delta \tag{19}$$

Near the surface in the boundary layer we assume that Λ is proportional to the distance normal to the surface, i.e.,

$$\Lambda = c_4 y \tag{20}$$

for $y \to 0$. An examination of the boundary conditions at $y = 0$ shows, however, that the constant c_4 must be related to c_1 so that only three parameters, c_1, c_2, and c_3, must be determined from experimental data.

To see the relationship between c_1 and c_4, we note that at the surface we must have

$$\mu\,\partial^2\langle u_i'u_j'\rangle/\partial y^2 = 2\mu\langle u_i'u_k'\rangle/\lambda^2 \tag{21}$$

Now $\langle u_i'u_j'\rangle$ must be zero at $y = 0$ and, since $\mu\partial\langle u_i'u_j'\rangle/\partial y$ represents the flux of $\langle u_i'u_j'\rangle$ through the surface $y = 0$ by viscous diffusion which is not physically possible, we must have $\partial\langle u_i'u_j'\rangle/\partial y = 0$ at $y = 0$. Thus we have, near $y = 0$,

$$\langle u_i'u_j'\rangle = a_2 y^2 + a_3 y^3 + \ldots \tag{22}$$

Substituting Eq. (22) into Eq. (21) results in

$$\lambda^2 = 2(a_2 y^2 + a_3 y^3 + \ldots)/(2a_2 + 6a_3 y) \tag{23}$$

or (unless $a_2 = 0$) for $y \to 0$

$$\lambda = y \tag{24}$$

Since $Re_\Lambda \to 0$, Eq. (18) can be written

$$\lambda^2 = \Lambda^2/c_1 \tag{25}$$

for small y. Thus, from Eqs. (20) and (24) we get

$$y^2 = c_4{}^2 y^2/c_1 \tag{26}$$

or

$$c_1 = c_4{}^2 \tag{27}$$

We have applied this model to the turbulent boundary layer on a flat plate and have compared the results with the standard data on such flows[6] in an attempt to determine the best values of c_1, c_2, and c_3. When this was originally done, the following values for the parameters were found

$$c_1 = 2.5;\quad c_2 = 0.125;\quad c_3 = 0.064 \tag{28}$$

Subsequently, an error was found in the computer program so that the values of the parameters given previously are not now considered to be those that will give an optimum fit to existing experimental data. They are, however, not far different from the optimum values. In the work that will be presented in the remainder of this paper, the values given in Eq. (28) were used. We believe, therefore, that the results we shall present are correct as to the general behavior of the atmosphere, although the exact numerical values of the various quantities computed will be altered somewhat when the new "optimum" values of the parameters that are presently being generated are inserted into our atmospheric programs.

Equations of Atmospheric Motion

Since the variation of density with altitude plays an important part in atmospheric motion, the equations above do not apply. The appropriate equations are generally written[7] in terms of the departure of the various physical quantities, T, ρ, p, and u_i, from the values they would have in an atmosphere at rest, namely, T_0, ρ_0, p_0, (which are functions of altitude only) and $u_{i_0} \equiv 0$. Thus, we define

$$\hat{T} = T - T_0;\quad \hat{\rho} = \rho - \rho_0;\quad \hat{p} = p - p_0;\quad \text{and}\quad \hat{u}_i = u_i \tag{29}$$

These quantities, in turn, are split into their mean and fluctuating parts, so that, for the rest of this paper, $\bar{T}$ stands for the mean of $\hat{T}$ and $T' = \hat{T} - \bar{T}$. The other variables are treated the same way.

With the assumption of small departures from equilibrium, the equations for the turbulent atmosphere are

$$\partial(\rho_0\bar{u}_i)/\partial t + (\rho_0\bar{u}^j\bar{u}_i)_{,j} = -\bar{p}_{,i} + [\mu g^{jm}(\bar{u}_{i,m}+\bar{u}_{m,i}) + \mu^*\delta_i{}^j\bar{u}_{,m}{}^m - \rho_0\langle u^{j\prime}u_i'\rangle]_{,j} + (\rho_0/T_0)\bar{T}\phi_{,i} \tag{30}$$

$$\partial(\rho_0 T)/\partial t + (\rho_0\bar{u}^j\bar{T})_{,j} = \mu g^{jm}\bar{T}_{,jm} - (\rho_0\langle u^{j\prime}T'\rangle)_{,j} \tag{31}$$

$$\bar{u}_{,j}{}^j = -(\bar{u}^j/\rho_0)\rho_{0,j} \tag{32}$$

$$u_{,j}{}^{j\prime} = -(u^{j\prime}/\rho_0)\rho_{0,j} \tag{33}$$

$$\begin{aligned}\rho_0\partial\langle u_i'u_j'\rangle/\partial t + \rho_0\bar{u}^m\langle u_i'u_j'\rangle_{,m} = {} & -\rho_0\langle u^{m\prime}u_j'\rangle\bar{u}_{i,m} - \rho_0\langle u^{m\prime}u_i'\rangle\bar{u}_{j,m} - (\rho_0\langle u^{m\prime}u_i'u_j'\rangle)_{,m} - \langle u_i'p'\rangle_{,j} - \langle u_j'p'\rangle_{,i} + \\ & \langle p'(u'_{i,j}+u'_{j,i})\rangle + \mu g^{mn}\langle u_i'u_j'\rangle_{,mn} - 2\mu g^{mn}\langle u'_{i,m}u'_{j,n}\rangle - \\ & (\mu+\mu^*)[\langle u^{m\prime}u_i'\rangle(\rho_{0,m}/\rho_0)_{,j} + \langle u^{m\prime}u_j'\rangle(\rho_{0,m}/\rho_0)_{,i} + \\ & (\rho_{0,m}/\rho_0)(\langle u_i'u_{,j}{}^{m\prime}\rangle + \langle u_j'u_{,i}{}^{m\prime}\rangle)] + \\ & (\rho_0/T_0)(\phi_{,i}\langle u_j'T'\rangle + \phi_{,j}\langle u_i'T'\rangle)\end{aligned} \tag{34}$$

$$\begin{aligned}[\rho_0\partial\langle u_i'T'\rangle/\partial t] + \rho_0\bar{u}^m\langle u_i'T'\rangle_{,m} = {} & -\rho_0\langle u^{m\prime}u_i'\rangle\bar{T}_{,m} - \rho_0\langle u^{m\prime}T'\rangle\bar{u}_{i,m} - (\rho_0\langle u^{m\prime}u_i'T'\rangle)_{,m} - \langle p'T'\rangle_{,i} + \langle p'T_{,i}'\rangle + \\ & \mu g^{mn}\langle u_i'T'\rangle_{,mn} - 2\mu g^{mn}\langle u'_{i,m}T_{,n}'\rangle - (\mu+\mu^*)\times \\ & [\langle u^{m\prime}T'\rangle(\rho_{0,m}/\rho_0)_{,i} + (\langle T'u_{,i}{}^{m\prime}\rangle\rho_{0,m}/\rho_0)] + \\ & (\rho_0/T_0)\phi_{,i}\langle T'^2\rangle\end{aligned} \tag{35}$$

$$[\rho_0\partial\langle T'^2\rangle/\partial t] + \rho_0\bar{u}^m\langle T'^2\rangle_{,m} = -2\rho_0\langle u^{m\prime}T'\rangle\bar{T}_{,m} - (\rho_0\langle u^{m\prime}T'^2\rangle)_{,m} + \mu g^{mn}\langle T'^2\rangle_{,mn} - 2\mu g^{mn}\langle T_{,m}'T_{,n}'\rangle \tag{36}$$

In these equations, we have taken the Prandtl number, $\mu c_p/k$, equal to one, but we have not made the usual assumption that $u_{,j}{}^{j\prime} = 0$ as is done in Ref. 7 and which, in general, restricts the motions which may be studied to those which are small in scale compared to the scale of the atmosphere.

If Eqs. (34, 35, and 36) are studied, one sees that, if the terms containing $\rho_0{}^{-1}\rho_{0,i}$ are dropped, the invariant modeling we have already carried out for boundary-layer flows may be taken over without the introduction of new parameters to the calculation of atmospheric motions. If we keep the terms containing $\rho_0{}^{-1}\rho_{0,i}$, we find that the modeling is changed slightly so as to satisfy the conservation law given in Eq. (33). The important point, however, is that no new parameters are introduced and the three parameters c_1, c_2, and c_3 discussed in the previous section are sufficient to determine the motion. We use the following modeling

$$\langle u^{m\prime}u_i'u_j'\rangle = -\Lambda q[g^{mn}\langle u_i'u_j'\rangle_{,n}+\langle u^{m\prime}u_i'\rangle_{,j}+\langle u^{m\prime}u_j'\rangle_{,i}] \tag{37}$$

$$\langle u^{m\prime}u_i'T'\rangle = -\Lambda q[g^{mn}\langle u_i'T'\rangle_{,n}+\langle u^{m\prime}T'\rangle_{,i}] \tag{38}$$

$$\langle u^{m\prime}T'^2\rangle = -\Lambda q g^{mn}\langle T'^2\rangle_{,n} \tag{39}$$

$$g^{mn}\langle u'_{i,m}u'_{j,n}\rangle = \langle u_i'u_j'\rangle/\lambda^2 \tag{40}$$

$$g^{mn}\langle u'_{i,m}T_{,n}'\rangle = \langle u_i'T'\rangle/\lambda^2 \tag{41}$$

$$g^{mn}\langle T_{,m}'T_{,n}'\rangle = \langle T'^2\rangle/\lambda^2 \tag{42}$$

$$\langle u_i'p'\rangle = -\rho_0\Lambda q\langle u^{m\prime}u_i'\rangle_{,m} \tag{43}$$

$$\langle p'T'\rangle = -\rho_0\Lambda q\langle u^{m\prime}T'\rangle_{,m} \tag{44}$$

$$\langle p'(u'_{i,j}+u'_{j,i})\rangle = (\rho_0 q/\Lambda)[g_{ij}(K/3)-\langle u_i'u_j'\rangle]- \langle u_i'p'\rangle \rho_{0,j}/\rho_0-\langle u_j'p'\rangle \rho_{0,i}/\rho_0 \quad (45)$$

$$\langle p'T_{,i}'\rangle = -(\rho_0 q/\Lambda)\langle u_i'T'\rangle \quad (46)$$

$$\langle u_i'u_{,j}^{m\prime}\rangle = -\langle u^{m\prime}u_i'\rangle \rho_{0,j}/\rho_0 \quad (47)$$

$$\langle T'u_{,i}^{m\prime}\rangle = -\langle u^{m\prime}T'\rangle \rho_{0,i}/\rho_0 \quad (48)$$

Equations (30–36), together with the modelings given above, now constitute the equations with which we propose to study the nature of turbulence generation in the atmosphere as a result of winds and thermal instabilities. Of course, this can only be done for very simple motions for which we will be able to identify the scale of the mean fields of $\bar{u}_i$ and $\bar{T}$. In what follows, we will apply the equations given in this section to a particularly simple example of atmospheric motion with the idea of demonstrating some of the behavior of atmospheric shear layers as indicated by our equations.

A Simple Case of Atmospheric Shear

In order to study the nature of atmospheric motion as described by the equations given above, we have selected what we consider to be the simplest motion that can give meaningful results. We consider an atmosphere in which the motion (u, v, w) in Cartesian coordinates (x, y, z) is such that $u = \bar{u}(z,t)$ and $\bar{v} = \bar{w} = 0$. Thus, we assume that the mean motion consists of only one component of velocity. That component of velocity is parallel to the ground and is only a function of time and altitude. In such a motion, symmetry requires that the correlations $\langle v'T'\rangle$, $\langle u'v'\rangle$, and $\langle v'w'\rangle$ be zero. To gain this amount of simplicity in the motion, it is necessary to invent a body force $X(z,t)$ which starts the motion. Actually, there is no such real force acting on the atmosphere but rather the atmosphere is driven by pressure and Coriolis forces. These forces, however, produce a far more complicated motion in the large than the one we have chosen. At the very least, one must, in such cases, consider a motion of the form $\bar{u}(z,t)$ and $\bar{w}(z,t)$. Nevertheless, these real motions can resemble locally the simple motions we shall study here. We believe that the essential features of atmospheric-turbulence production will be demonstrated by our somewhat fictitious atmospheric model while retaining a relatively simple set of governing equations. For the motion we have just described, the equations given in the previous section become

$$\rho_0\,\partial\bar{u}/\partial t = (\mu\,\partial^2\bar{u}/\partial z^2)-[\partial(\rho_0\langle u'w'\rangle)/\partial z]+X(z,t) \quad (49)$$

$$\rho_0\,\partial\bar{T}/\partial t = (\mu\,\partial^2\bar{T}/\partial z^2)-[\partial(\rho_0\langle w'T'\rangle)/\partial z] \quad (50)$$

$$\partial\bar{p}/\partial z = -(\partial/\partial z)(\rho_0\langle w'w'\rangle)+(\rho_0 g/T_0)\bar{T} \quad (51)$$

$$\frac{\partial\langle u'u'\rangle}{\partial t} = -2\langle u'w'\rangle\frac{\partial\bar{u}}{\partial z}+\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 q\Lambda\frac{\partial\langle u'u'\rangle}{\partial z}\right)- \frac{q}{\Lambda}\left(\langle u'u'\rangle-\frac{K}{3}\right)+\frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle u'u'\rangle-\frac{2\mu}{\rho_0}\frac{\langle u'u'\rangle}{\lambda^2} \quad (52)$$

$$\frac{\partial\langle v'v'\rangle}{\partial t} = \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 q\Lambda\frac{\partial\langle v'v'\rangle}{\partial z}\right)-\frac{q}{\Lambda}\left(\langle v'v'\rangle-\frac{K}{3}\right)+ \frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle v'v'\rangle-\frac{2\mu}{\rho_0}\frac{\langle v'v'\rangle}{\lambda^2} \quad (53)$$

$$\frac{\partial\langle w'w'\rangle}{\partial t} = \frac{3}{\rho_0}\frac{\partial}{\partial z}\left(\rho^0 q\Lambda\frac{\partial\langle w'w'\rangle}{\partial z}\right)-\frac{q}{\Lambda}\left(\langle w'w'\rangle-\frac{K}{3}\right)+ 2\sigma\Lambda q\frac{\partial\langle w'w'\rangle}{\partial z}+\frac{2}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 q\Lambda\frac{\partial\langle w'w'\rangle}{\partial z}\right)+ \frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle w'w'\rangle-\frac{2\mu}{\rho_0}\frac{\langle w'w'\rangle}{\lambda^2}-\frac{2(\mu+\mu^*)}{\rho_0}\times \left[\langle w'w'\rangle\left(\frac{\partial\sigma}{\partial z}-\sigma^2\right)\right]+\frac{2g}{T_0}\langle w'T'\rangle \quad (54)$$

$$\frac{\partial\langle u'w'\rangle}{\partial t} = -\langle w'w'\rangle\frac{\partial\bar{u}}{\partial z}+\frac{2}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 q\Lambda\frac{\partial\langle u'w'\rangle}{\partial z}\right)+ \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 q\Lambda\frac{\partial\langle u'w'\rangle}{\partial z}\right)-\frac{q}{\Lambda}\langle u'w'\rangle+\sigma\Lambda q\frac{\partial\langle u'w'\rangle}{\partial z}+ \frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle u'w'\rangle-\frac{2\mu}{\rho_0}\frac{\langle u'w'\rangle}{\lambda^2}-\frac{(\mu+\mu^*)}{\rho_0}\times \left[\langle u'w'\rangle\left(\frac{\partial\sigma}{\partial z}-\sigma^2\right)\right]+\frac{g}{T_0}\langle u'T'\rangle \quad (55)$$

$$\frac{\partial\langle u'T'\rangle}{\partial t} = -\langle u'w'\rangle\frac{\partial\bar{T}}{\partial z}-\langle w'T'\rangle\frac{\partial\bar{u}}{\partial z}+\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 q\Lambda\frac{\partial\langle u'T'\rangle}{\partial z}\right)- \frac{q}{\Lambda}\langle u'T'\rangle+\frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle u'T'\rangle-\frac{2\mu}{\rho_0}\frac{\langle u'T'\rangle}{\lambda^2} \quad (56)$$

$$\frac{\partial\langle w'T'\rangle}{\partial t} = -\langle w'w'\rangle\frac{\partial\bar{T}}{\partial z}+\frac{2}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 q\Lambda\frac{\partial\langle w'T'\rangle}{\partial z}\right)+ \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 q\Lambda\frac{\partial\langle w'T'\rangle}{\partial z}\right)-\frac{q}{\Lambda}\langle w'T'\rangle+\frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle w'T'\rangle- \frac{2\mu}{\rho_0}\frac{\langle w'T'\rangle}{\lambda^2}-\frac{(\mu+\mu^*)}{\rho_0}\left[\langle w'T'\rangle\left(\frac{\partial\sigma}{\partial z}-\sigma^2\right)\right]+\frac{g}{T_0}\langle T'^2\rangle \quad (57)$$

$$\frac{\partial\langle T'^2\rangle}{\partial t} = -2\langle w'T'\rangle\frac{\partial\bar{T}}{\partial z}+\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 q\Lambda\frac{\partial\langle T'^2\rangle}{\partial z}\right)+ \frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle T'^2\rangle-\left(\frac{2\mu}{\rho_0}\frac{\langle T'^2\rangle}{\lambda^2}\right) \quad (58)$$

From our previous work on invariant modeling, we will keep the following results

$$\lambda^2 = \Lambda^2/(2.5+0.125\,Re_\Lambda) \quad (59)$$

$$\Lambda = 0.064\delta \quad (60)$$

In this last equation, we will assume that δ is some appropriately chosen scale of the mean motion which will be akin to the boundary-layer thickness of our previous studies.

In the present study, we do not consider any interaction of the turbulence that is produced with the ground since we will, in general, be studying the generation of shear layers at altitude. However, if we were to do so, we would choose

$$\Lambda = (2.5)^{1/2}z \quad (61)$$

in the region from the ground to the point where

$$z = [0.064/(2.5)^{1/2}]\,\delta \quad (62)$$

Initial Computations

To date we have made only a few computations based on the scheme laid out in the preceding section. These initial computations represent the simplified cases which were used to check the program and are not meant to represent any particular physical situation. As may be seen by an examination of Eqs. (49–57), if we neglect the terms containing σ, which amounts to assuming the usual equations of atmospheric motion for which the scale of the motion is small compared to the scale of the atmosphere, and if we take, consistent with this assumption, ρ_0 equal to a constant, the solution of the equations will exhibit certain symmetries with respect to symmetric initial inputs that are useful in checking the program. The four problems we shall discuss are solutions of Eqs. (49–57) for the case when ρ_0 is constant.

The atmosphere is assumed to have initially a 4000-ft thick band of turbulence that is isotropic with $\langle u'^2\rangle = \langle v'^2\rangle = \langle w'^2\rangle = 1$ (fps)^2. The band is centered at an altitude of 20,000 ft although the actual value of the altitude has no bearing on the problem due to the assumption that ρ_0 is constant. (The initial sharp edges of the turbulent band are soon smoothed off by the action of the diffusion and dissipation terms of the equations.) The atmosphere is initially at rest, i.e., at $t = 0, \bar{u} = 0$. For a time, a body force acts on the atmosphere, creating a mean motion. The body force in the calculations we present here was taken to be of the form (Fig. 1)

$$X(z,t) = c(1-\xi^2)^2 \quad (63)$$

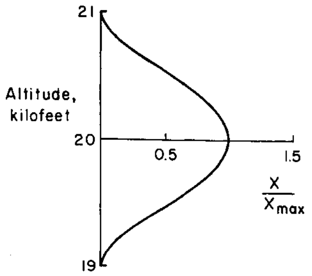


Fig. 1 Shape of the mean-velocity forcing function.

where

$$\xi = (z-20{,}000)/1000 \tag{64}$$

for

$$|\xi| < 1 \tag{65}$$

In Eq. (64), z is an actual altitude in feet. Outside the region $|\xi| < 1$, the driving force is taken to be zero. Thus, the forcing function is such that it produces a mean shear layer some 2000 ft thick. In the particular examples which we consider here, this forcing function is such that in the absence of any turbulence or viscosity the centerline velocity would increase to 30 fps in 3000 sec. After 3000 sec, it was assumed to vanish and the atmosphere left to dissipate the motion by turbulence and viscous diffusion.

Four cases of initial mean temperature distribution were considered (Fig. 2).

Case I: neutrally stable atmosphere $\bar{T}(z,0) = 0$.

Case II: unstable–stable–unstable atmosphere given by $\bar{T}(z,0) = (135/32)\xi(2-|\xi|)^2$ for $|\xi| < 2$ and zero elsewhere. Notice that this initial temperature is spread over 4000 ft rather than 2000 ft as was the case for the forcing function. The maximum temperature difference is 10°R.

Case III: stable–unstable–stable atmosphere given by $\bar{T}(z,0) = -(135/32)\xi(2-|\xi|)^2$ in the region $|\xi| < 2$ and zero elsewhere.

Case IV: stable–unstable atmosphere given by $\bar{T}(z,0) = 10(1-\eta^2)^2$ where

$$\eta = \tfrac{1}{2}\xi = (z-20{,}000)/2000 \tag{66}$$

for $|\eta| < 1$ and zero elsewhere. This formula again gives a maximum temperature difference of 10°R.

The other initial conditions that have been used for these computations are

$$\langle u'w'\rangle(z,0) = 0; \quad \langle w'T'\rangle(z,0) = 0; \quad \text{and} \quad \langle T'^2\rangle(z,0) = 0$$

The scale Λ for all calculations was taken to be the length defined by 0.064 times the breadth of the mean shear layer as based on the distance between the points where the mean velocity falls to half its maximum value.

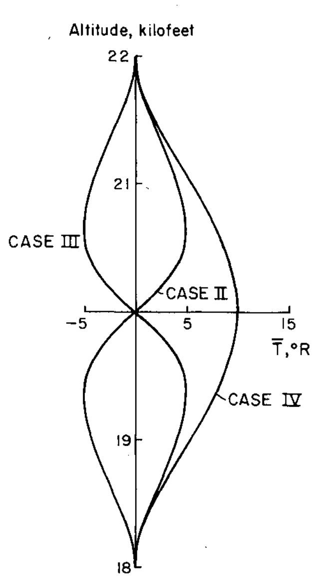


Fig. 2 Initial temperature profiles.

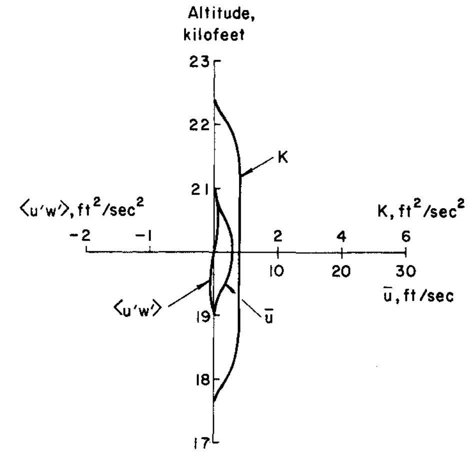


Fig. 3 $\bar{u}$, $\langle u'w'\rangle$, **and** $K = \langle u'^2\rangle + \langle v'^2\rangle + \langle w'^2\rangle$ **for a neutrally stable atmosphere; Case I at 300 sec.**

Results

Case I: Neutrally stable atmosphere

In Figs. 3, 4, and 5, we show the calculated values of the mean velocity $\bar{u}$, $K = \langle u'^2\rangle + \langle v'^2\rangle + \langle w'^2\rangle$, and $\langle u'w'\rangle$ for three times, 300, 3000, and 6000 sec, respectively. The first figure shows the very start of the development of the mean motion. At this time, very little turbulence has been created by the mean motion. Most of the turbulent energy that is seen is left over from the decaying initial turbulent layer. In Fig. 4, which shows the motion at the end of the application of the forcing function, the mean velocity is a maximum. The production of turbulent energy as a result of the deformation $\partial\bar{u}/\partial z$ is evident. The magnitudes of K and $\langle u'w'\rangle$ that are calculated are consistent with the results of tests on freejets in the laboratory. Figure 5 shows the motion after decay has set in. The mean velocity has spread, and the centerline value of K has started to fill in by diffusion.

Another picture of the development of this motion can be had by plotting the maxima of the various correlations determined at any time against time. This is done in Fig. 6. Here we see the initial decay of the turbulent layer with which we started, followed by the production of the various correlations as a result of the interaction between this turbulence and the mean flow. Once the forcing function ceases to increase the mean velocity, the jet that has been formed decays in the typical self-similar manner of laboratory jets.

Case II: Unstable-stable-unstable atmosphere

In Fig. 7, we show the mean-velocity profile at 3000 sec for the production of a shear layer in an unstable–stable–unstable

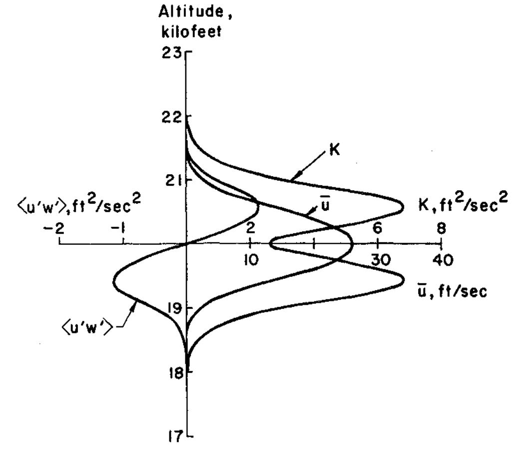


Fig. 4 $\bar{u}$, $\langle u'w'\rangle$, **and** K **for Case I at 3000 sec.**

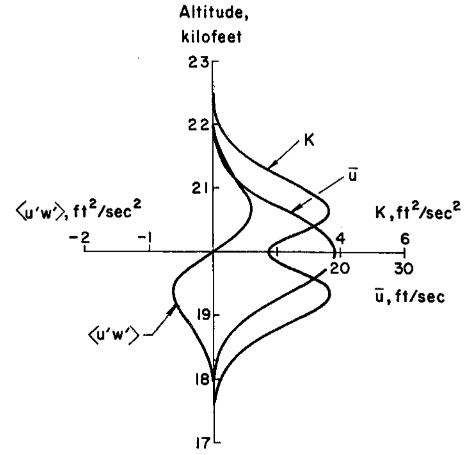


Fig. 5 $\bar{u}$, $\langle w'w' \rangle$, **and** K **for Case I at 6000 sec.**

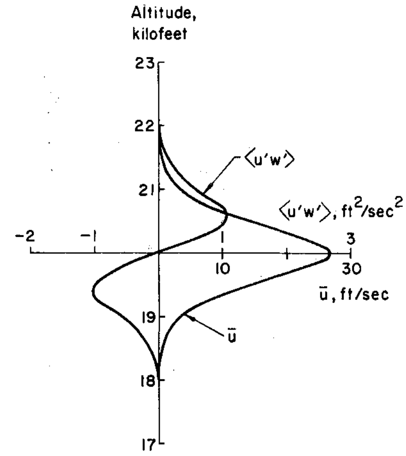


Fig. 7 $\bar{u}$ **and** $\langle u'w' \rangle$ **for Case II at 3000 sec.**

atmosphere. We note from a comparison of this figure with Fig. 4 (and the result is generally true) that at the maximum of the mean velocity, the mean velocity and shear $-\langle u'w' \rangle$ are changed somewhat but not drastically by the effect of the temperature profile. As may be seen in Fig. 8, where we have plotted K and $\langle T'^2 \rangle$ for both 300 and 3000 sec, the development of the turbulence is significantly altered. At small times, the contribution of the vertical component of velocity to the turbulent energy is large and is due to the basic thermal instability. The maximum value of K at this time is well outside the region of normal production by $\partial \bar{u}/\partial z$. As time progresses, however, the production of turbulent energy by $\partial \bar{u}/\partial z$ becomes large, and the maximum of K occurs near the maximum of $\partial \bar{u}/\partial z$ as may be seen from the K curve for 3000 sec.

In Fig. 9, we show the behavior of the heat-flux correlation $\langle w'T' \rangle$ and the mean temperature for times of 300 and 3000 sec. Here we note the very large values of $\langle w'T' \rangle$ in regions of $\partial \bar{T}/\partial z < 0$ and the very small values of $\langle w'T' \rangle$ in regions of $\partial \bar{T}/\partial z > 0$. The values of $|\partial \bar{T}/\partial z|$ are not very different, and the level of turbulence K at times greater than 3000 sec cannot account for this difference. That is to say, none of the usual eddy diffusivity models could account for this behavior and the very persistent nature of the inversion that has been calculated here. A close inspection of the solution in the neighborhood of $z = 20{,}000$ (the center of the inversion) indicates that at small times (t less than approximately 300 sec), the solution for $\langle w'T' \rangle$ is oscillatory. This strange behavior of the solution is not a quirk of the program or the modeling procedures that are used but is a result of the basic nature of the production equations when the atmosphere is stable, as will be discussed below.

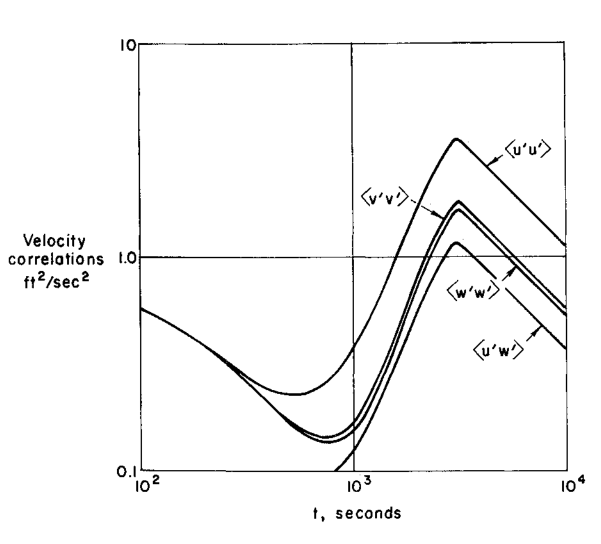


Fig. 6 Maxima of velocity correlations; Case I.

In Fig. 10, we have again plotted the behavior of the maxima of the velocity correlations as a function of time. Note first the general production of turbulence by thermal instability at small times. Near the maximum of the mean velocity, the production of turbulence resembles that of the neutral atmosphere. Once the forcing function disappears, the mean motion dissipates and the thermal instabilities in the outer regions of the motion take over. In this case they just about balance dissipation as time proceeds. This is due to the increase in the dissipation scale λ in our calculations as time proceeds.

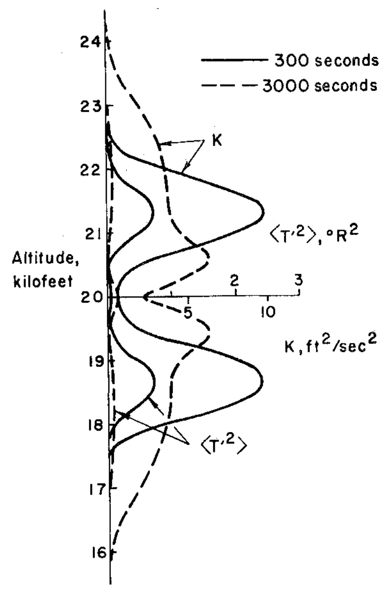
Case III: Stable-unstable-stable atmosphere

In Fig. 11, we show the mean velocity at 3000 sec for the production of a shear layer in a stable–unstable–stable atmosphere. Note again that this is not much changed from the two previous cases although there has been slightly more diffusion of the mean velocity profile.

In Fig. 12, we again see a marked effect of the mean-temperature profile on the turbulence production. In this case, the thermal instability at small times produces a great deal of turbulence in the unstable layer trapped between two stable layers. Only towards the maximum of the mean motion does the shear-produced turbulence make its presence felt.

Figure 13 is a plot of the behavior of $\langle w'T' \rangle$ and $\bar{T}$. Here we see again that very large turbulent heat fluxes are developed in the unstabe region $\partial \bar{T}/\partial z < 0$, while it has been virtually impossible to produce any heat flux in the region where $\partial \bar{T}/\partial z > 0$. This is another example of the difficulty of getting rid of inver-

Fig. 8 $\langle T'^2 \rangle$ **and** K **for Case II at 3000 sec.**



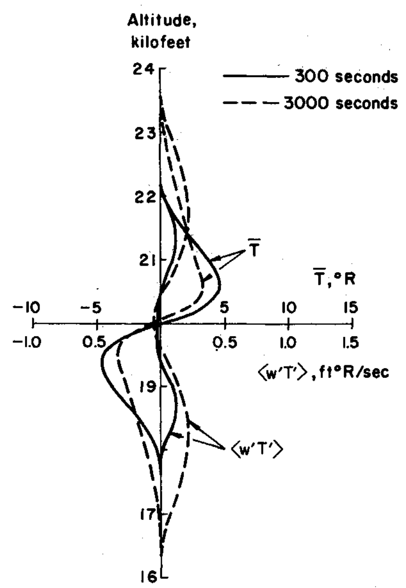


Fig. 9 $\bar{T}$ and $\langle w'T' \rangle$ for Case II at 300 and 3000 sec.

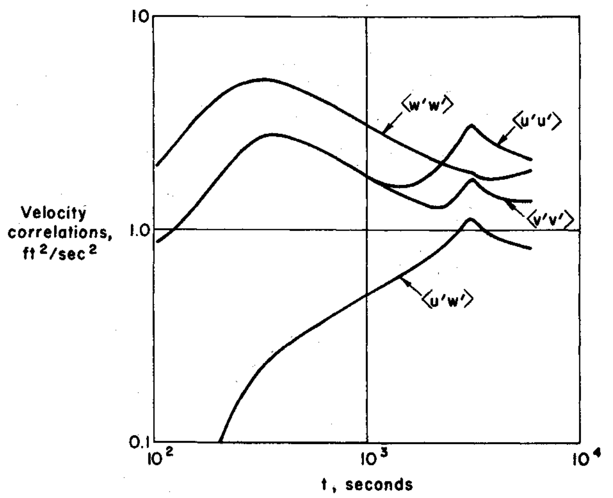


Fig. 10 Maxima of velocity correlations; Case II.

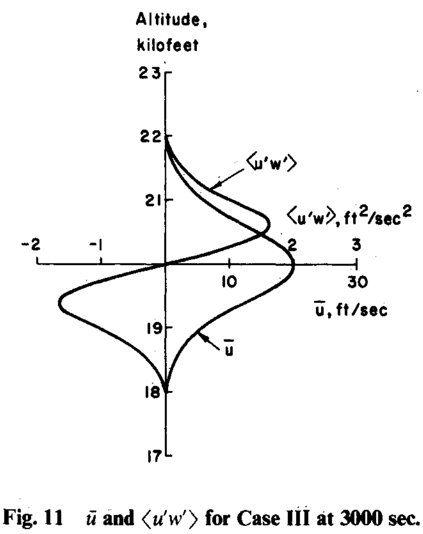


Fig. 11 $\bar{u}$ and $\langle u'w' \rangle$ for Case III at 3000 sec.

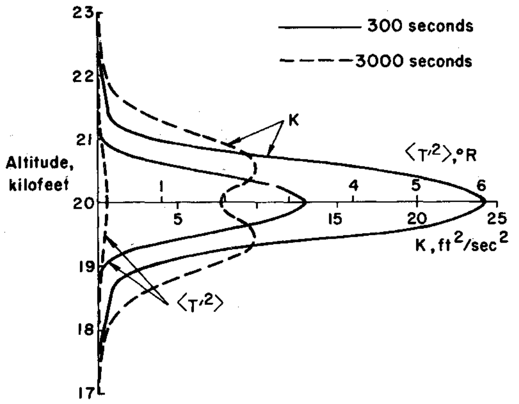


Fig. 12 $\langle T'^2 \rangle$ and K for Case III at 300 and 3000 sec.

sions. Again this reluctance of nature to establish a heat flux when $\partial \bar{T}/\partial z > 0$ is a basic characteristic of the atmosphere and cannot be described by any simple eddy-diffusivity model.

In Fig. 14 we show the behavior of the maxima of the velocity correlations as a function of time for this case of shear-layer formation. Note that the effect of shear-generated turbulence makes itself evident only when the mean flow is near its maximum velocity.

Case IV: Stable-unstable atmosphere

In Fig. 15, we have plotted the mean-velocity profile and the profile of $\langle u'w' \rangle$ for a time of 3000 sec. Here we again note the similarity of this profile to all the previous cases.

In Fig. 16, we show the behavior of $\langle T'^2 \rangle$ and K for $t = 3000$ sec. In this case we note that there are two extrema of K. Both are close to the point where $\partial \bar{u}/\partial z$ is a maximum, as might be expected. The production of turbulent energy on the unstable side exceeds that on the stable side but not by as much as we had expected prior to making these calculations.

In contrast to the behavior of the turbulent energy, we see from Fig. 17 that the production of $\langle w'T' \rangle$ is vastly different in the stable and in the unstable regions. We note again that it is extremely difficult to produce any $\langle w'T' \rangle$ in a stable region, so that the temperature inversion ($\partial \bar{T}/\partial z > 0$) is extremely persistent. Again, no presently used eddy-diffusivity model would predict the result we have shown.

In Fig. 18, we show the behavior of the maxima of the velocity correlations as a functon of time. We note again the initial production of turbulence by means of thermal instability, the modification of this by the action of shear-generated turbulence near the time of maximum mean velocity, and the final phase of the motion when the production of turbulence by

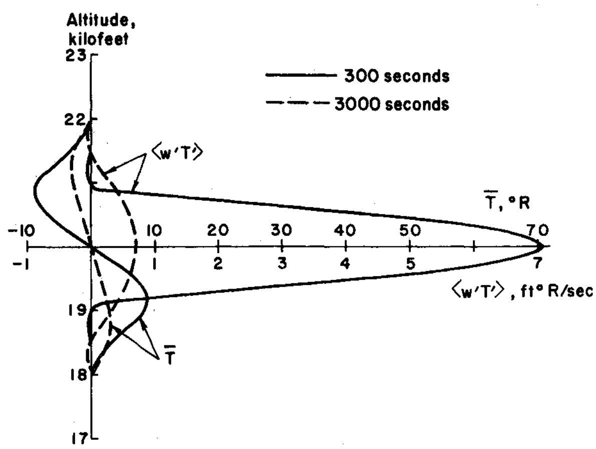


Fig. 13 $\bar{T}$ and $\langle w'T' \rangle$ for Case III at 300 and 3000 sec.

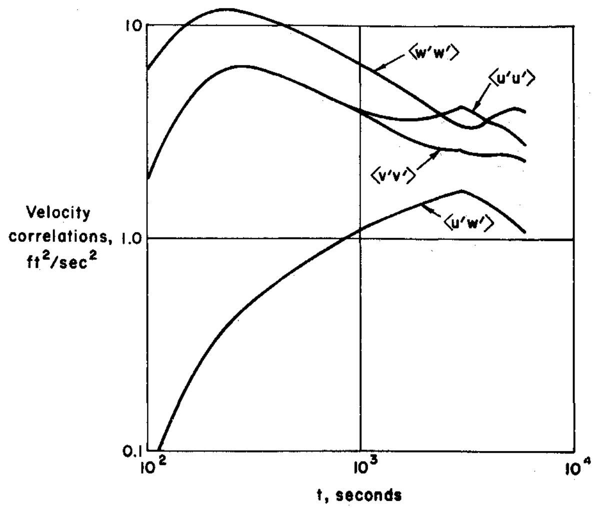


Fig. 14 Maxima of velocity correlations; Case III.

thermal instability in the decaying thermal layer is just about balanced by dissipation so that a rather constant level of turbulent energy results.

Discussion of Results

The most interesting aspect of the results we have just presented is the radically different behavior of the heat flux correlation $\langle w'T'\rangle$ depending on whether $\partial\bar{T}/\partial z$ is greater or less than zero. The explanation for this behavior lies in the equations and may be exhibited as follows.

Consider a flow in which there is no mean motion and in which all quantities except $\bar{T}$ are independent of z. In this case, the pertinent equations for the production of correlations obtained from Eqs. (54, 57, and 58) are

$$\frac{\partial\langle w'w'\rangle}{\partial t}=\frac{2g}{T_0}\langle w'T'\rangle-\frac{q}{\Lambda}\left(\langle w'w'\rangle-\frac{K}{3}\right)-\frac{2\mu}{\rho_0}\frac{\langle w'w'\rangle}{\lambda^2} \tag{67}$$

$$\frac{\partial\langle w'T'\rangle}{\partial t}=-\langle w'w'\rangle\frac{\partial\bar{T}}{\partial z}+\frac{g}{T_0}\langle T'^2\rangle-\frac{q}{\Lambda}\langle w'T'\rangle-\frac{2\mu}{\rho_0}\langle w'T'\rangle \tag{68}$$

$$\frac{\partial\langle T'^2\rangle}{\partial t}=-2\langle w'T'\rangle\frac{\partial\bar{T}}{\partial z}-\frac{2\mu}{\rho_0}\frac{\langle T'^2\rangle}{\lambda^2} \tag{69}$$

If we consider only the production terms and neglect the tendency-towards-isotropy terms and dissipation terms in these equations, we have

$$\partial\langle w'w'\rangle/\partial t=(2g/T_0)\langle w'T'\rangle \tag{70}$$

$$\partial\langle w'T'\rangle/\partial t=-\langle w'w'\rangle\,\partial\bar{T}/\partial z+(g/T_0)\langle T'^2\rangle \tag{71}$$

$$\partial\langle T'^2\rangle/\partial t=-2\langle w'T'\rangle\,\partial\bar{T}/\partial z \tag{72}$$

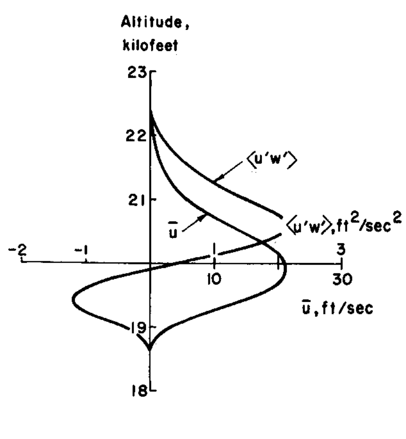


Fig. 15 $\bar{u}$ and $\langle u'w'\rangle$ for Case IV at 3000 sec.

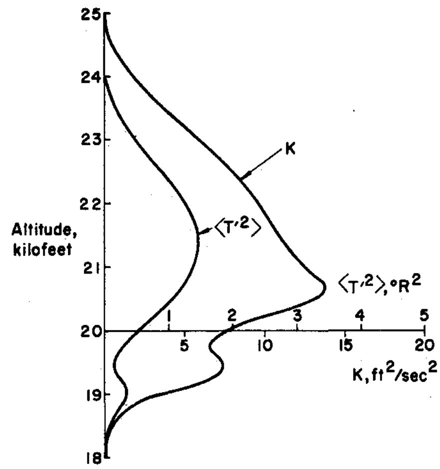


Fig. 16 $\langle T'^2\rangle$ and K for Case IV at 3000 sec.

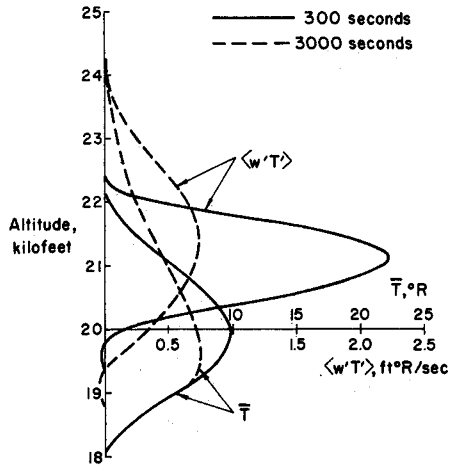


Fig. 17 $\bar{T}$ and $\langle w'T'\rangle$ for Case IV at 300 and 3000 sec.

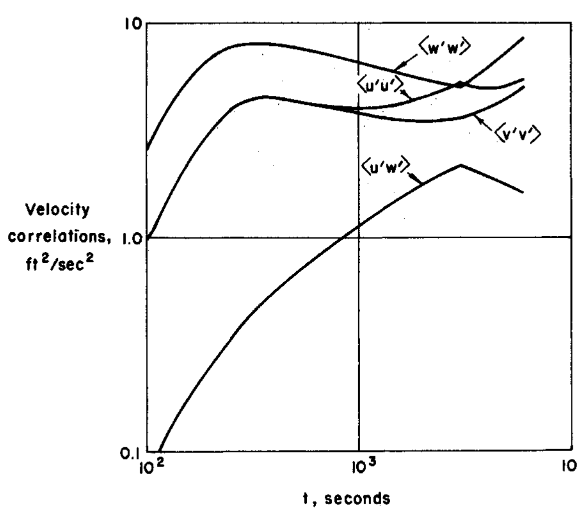


Fig. 18 Maxima of velocity correlations; Case IV.

We now assume (on the basis of experience with existing calculations) that the rate of change of $\partial\bar{T}/\partial z$ is very slow compared to the rate of change of the second-order correlation. In this case we can assume, to first order, that $\partial\bar{T}/\partial z$ is a constant and eliminate $\langle w'w'\rangle$ and $\langle T'^2\rangle$ from Eq. (71) by differentiating that equation with respect to time and using Eqs. (70) and (72). The result is

$$\partial^2\langle w'T'\rangle/\partial t^2 = -(4g/T_0)(\partial\bar{T}/\partial z)\langle w'T'\rangle \tag{73}$$

Examination of this equation shows that when $\partial\bar{T}/\partial z < 0$, there is an exponential development of $\langle w'T'\rangle$. However, when the atmosphere is stable, i.e., $\partial\bar{T}/\partial z > 0$, the heat-flux correlation $\langle w'T'\rangle$ is oscillatory with frequency

$$\omega = 2[(g/T_0)\partial\bar{T}/\partial z]^{1/2} \tag{74}$$

In fact the general solution for Eqs. (70–72) may be written

$$\langle w'T'\rangle = A\sin\omega t + B\cos\omega t \tag{75}$$

$$\langle w'w'\rangle = [\omega/(2\partial\bar{T}/\partial z)][-A\cos\omega t + B\sin\omega t] + C \tag{76}$$

$$\langle T'^2\rangle = (2/\omega)(\partial\bar{T}/\partial z)[A\cos\omega t - B\sin\omega t] + (T_0/g)(\partial\bar{T}/\partial z)C \tag{77}$$

When the general case in which both the tendency-towards-isotropy terms and dissipation are considered, i.e., Eqs. (67–69), there results a damped oscillaton of the quantities $\langle w'T'\rangle$, $\langle w'w'\rangle$, and $\langle T'^2\rangle$. In fact, if Λ is sufficiently large compared to λ so that only dissipation need be considered, the solution becomes $\exp[-2\mu t/(\rho_0\lambda^2)]$ times each of the expressions given above.

A close examination of all our numerical results shows this oscillatory behavior at the frequency given above during the early phases of each run in the region where $\partial\bar{T}/\partial z > 0$.

The upshot of all this discussion is that the basic equations of atmospheric motion indicate a vast difference between the behavior of $\langle w'T'\rangle$ when $\partial\bar{T}/\partial z < 0$ and when $\partial\bar{T}/\partial z > 0$. In the unstable case, a large heat flux is established quite rapidly. In the stable case, the heat flux tends to oscillate about a value that is very small. This results in an extraordinary persistence for temperature inversions in the atmosphere. It also means that the idea of a simple eddy diffusivity in the atmosphere is completely wrong and will, when used in calculations, result in far too rapid a dissipation of any inversions.

Conclusions and Recommendations

We have presented the first results using the method of invariant modeling to calculate the behavior of atmospheric shear layers. We realize we have only scratched the surface; much remains to be done. In particular, we are working towards a better method for putting the scale length Λ into our equations. We would eventually like to have an explicit equation for Λ which could be carried along as an extra simultaneous equation to be solved. For certain cases, namely, the production of thermal turbulence, this is probably more important than in the case of shear-driven turbulence. In this connection, mention should be made of the recent work of Harlow et al.[8,9] which is closely related to our own studies.

Although the authors recognize that much still needs to be done to finalize a general method for predicting wind-driven atmospheric motion, we feel that the method of invariant modeling represents a powerful basic tool with which to approach this problem. It seems clear from our studies to date that the eddy-diffusivity concept should be discarded as a means by which the gas dynamicist attacks the problem of turbulent motions in the atmosphere.

In regard to future applications of the method we have just presented, it should be mentioned that equations similar to those considered here can be written for correlations of fluctuations in species concentration. Thus, application of this method to the convection and diffusion of gaseous pollutants in the atmosphere is straightforward. In particular, the effect of a stable lapse rate on the development of turbulent mass transport is similar to that found for the turbulent transport of heat in this study.

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